Investigation of logarithmic spirals in nature by means of dynamic geometry and computer algebra systems

Roman Hašek
e-mail: hasek@pf.jcu.cz
Department of mathematics
University of South Bohemia
České Budějovice, 371 15
Czech Republic

Abstract

The aim of the paper is to show how we can combine the pupil's knowledge with the computational and graphical power of computer algebra and dynamic geometry systems to model some real-world phenomena. The software can serve as a bridge over mathematically or geometrically complex parts of such a process and enable us to bring selected real-world problems into mathematics classes. We will illustrate it by means of several examples, all more or less dealing with the exponential dependence, especially its manifestation through the shape of the logarithmic spiral. The examples will be solved with the use of the free open source software GeoGebra and wxMaxima.

1 Introduction

The aim of this paper is to show, via selected particular examples, a way of using a dynamic geometry software (DGS) together with a computer algebra system (CAS) to produce geometrical and mathematical models of various real-world phenomena.

The solving of real-world problems and the analysis of real-world phenomena in mathematics lessons meet the aims of the constructivist teaching method [14], the important role of which is recognized in mathematics education [7] and can be concisely characterized by the next quotation from [4]: "The constructivist approach to the teaching of mathematics is characterized by the active formation of elements of mathematics in the mental world of a child. A background for such construction can be a question or a real-world problem from the world of nature, technique or mathematics itself, according to the nature of the child."

There are various real-world problems or phenomena to be found around us. Of course, in contrast to artificial problems from textbooks, calculations corresponding to real-world problems usually do not have neat results. They often do not give integral values or lead to the higher-degree polynomial or transcendental equations that require numerical solutions. Sometimes we have to do exhausting

repetitive computations. We also often face the necessity of using some higher level of mathematical knowledge on the way to a solution. Then, well selected software, suitably used, enables a student, with the guidance of his/her teacher, to overcome such barriers and to arrive at a final result. The software is used as the tool of investigation into the given problem. Such a process will be illustrated by several examples, all more or less dealing with the exponential dependence, especially its manifestation through the shape of the logarithmic spiral, that will be solved by means of the free open source software GeoGebra [17] and wxMaxima [19].

The examples are selected from problems that we practice with our students of mathematics teaching in the pre-service teacher training programs or with mathematics teachers within the scope of in-service teacher training programs at the University of South Bohemia. The examples have proved to be a good means of illustrating the role of Euler's number e or the exponential dependence in the description of some real-world processes. They facilitate students' understanding of the features of the exponential function and the phenomenon of continuous growth or decay.

The reasons for the selection of GeoGebra and wxMaxima are partly given by the technical circumstances, because we use them in various courses at the faculty, but in the main we are considering their suitability for performing the intended activities. GeoGebra especially, thanks to its unique combination of spreadsheet, drawing pad and algebraic representation of objects, represents a powerful tool that enables us to draw graphs and curves, investigate their qualities and simply perform numerical iterative computation and to plot its results.

One of the methods used is based on the analysis of photos placed on the background of the GeoGebra geometric desktop. This highly motivating method, which compares the real curve on a photo with the curve drawn in the program, was used in [2]. We will also apply the GeoGebra Spreadsheet to conduct repetitive numerical computations. This method is inspired by the numerical methods in [3].

The description of the problem solving is not absolutely in-depth in the paper but it should be sufficient to enable a reader to solve the examples by himself or to inspire him/her to try similar investigations with a computer. It is not necessary to keep the given software. Most of the presented activities can be adapted to different software if it provides the same or similar tools.

2 Euler's number

First, we will take advantage of the combination of a spreadsheet with the symbolic capabilities of CAS to illustrate the importance of Euler's number e with a couple of examples from the world of finance. These problems are well known and have been solved in various publications but without regard to their prevalence it is worth solving them with students to illustrate briefly to them the connection of Euler's number to the phenomenon of continuous growth or decay.

In various non-fiction books devoted to mathematics, see for example [6, 8, 15], a uniqueness of Euler's number e is illustrated by the next example.

Example: *Consider 1 EUR deposited for one year in a saving account that pays 100 % annual interest rate. What will be the relation between the balance at the end of the year and the number of compounding periods per year (i.e. compounding frequency)? Take into account various possibilities of compounding frequency, from one, two, three or more to infinitely many (so-called continuous compounding) compounding periods per year.*

The case of the indefinite increase of the number m of compounding periods per year can be written mathematically as the limit of the expression $\left(1 + \frac{1}{m}\right)^m$ as m approaches infinity, which is well known from introductory courses to Calculus as defining the Euler's number e:

$$\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = e. \tag{1}$$

However, we can solve the problem without the limit simply by using a spreadsheet to evaluate the expression $\left(1 + \frac{1}{m}\right)^m$ for progression of the increasing values of m and to plot its values approaching the Euler's number. See the result of the GeoGebra utilization [20] in Figure 1 a). If students have not yet undertaken any calculus course we can confirm our experimental result concerning the limit (1) by means of a CAS, for example wxMaxima [21]:

```
(%i1)  'limit((1+1/m)^m,m,infinity)=limit((1+1/m)^m,m,infinity);
```

$$(\%o1) \quad \lim_{m \to infinity} \left(\frac{1}{m} + 1\right)^m = e$$

Then, knowing this feature of e, we can proceed with solving the following more practical financial problem.

Problem: *What maximum financial return can we acquire from an account if its compounding frequency is variable? Consider 1 000 EUR initial principal invested in an account that pays 8 % annual interest rate, compounded $m-$times per year.*

According to the solution of the previous problem we know that the maximum profit is earned with continuous compounding. Then, one year's maximum account balance corresponding to the principal of 1 000 EUR is given by the formula

$$P_1 = \lim_{m \to \infty} 1000 \left(1 + \frac{0.08}{m}\right)^m = 1000 e^{0.08} \approx 1083 \, EUR. \tag{2}$$

Generally, the maximum balance P_n from the principal P_0 after n years of saving with the annual interest rate of $100 \cdot i$ percent (i.e. i is a decimal number) is defined by

$$P_n = \lim_{m \to \infty} P_0 \left(1 + \frac{i}{m}\right)^{m \cdot n} = P_0 e^{n \cdot i}. \tag{3}$$

Graphical representation of (3) after assigning the given parameters' values ($P_0 = 1000$, $i = 0.08$) is shown in Figure 1 b).

And again, if students do not have sufficient calculus background wxMaxima [23] can help:

```
(%i1)  P[n]=limit(P[0]*(1+i/m)^(m*n),m,infinity);
```

$$(\%o1) \quad P_n = P_0 \, e^{i \, n}$$

The significance of the two above-mentioned examples dealing with the number e exceeds the range of financial matters. They illustrate a unique importance of *e* for the mathematical description of all such processes, the common typical feature of which is a growth or a decrease in some characteristics.

a) A one year saving, [20]   b) Continuous compounding, [22]
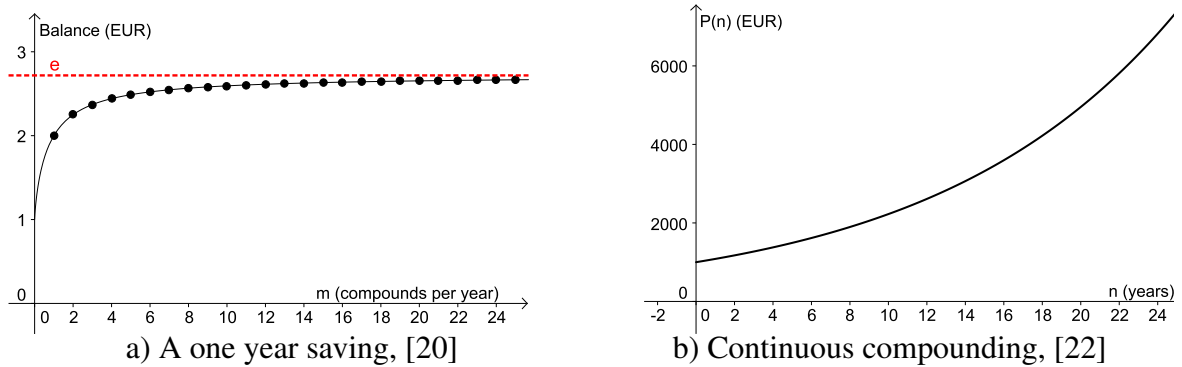
Figure 1: Euler's number in financial issues (Plots produced with [17].)

Specifically, we consider such phenomena of which the dependent quantity growth rates (e.g. the rate of growth of the balance) are directly proportional to the instant values of these quantities (e.g. instant amount of the balance). To give more examples we can mention the growth of a bacterial colony in some food, decay of a radioactive isotope in a mineral sample or the process of an object cooling under the influence of a lower ambient temperature (so-called Newton's law of cooling). Generally we speak about an exponential growth or decay model [11]. Mathematical descriptions of these processes through differential equations are parts of university introductory calculus courses. However, it is not necessary to have knowledge of the solving of differential equations to check the relationship of all such formulas to (3). We can benefit from the symbolic capabilities of wxMaxima. All we have to do is to explain to our students that the process described in the first sentence of this paragraph can be expressed by the formula

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = kQ(t), \tag{4}$$

where $Q(t)$ is the instant amount of a quantity in our view in time t, $\dfrac{\mathrm{d}Q(t)}{\mathrm{d}t}$ is the instant rate of its growth or decay and k is the real constant given by the initial conditions. The rest of the explanation is provided by wxMaxima [24]:

```
(%i1)  diff(Q(t),t)=k*Q(t);
```

(%o1) $\dfrac{d}{d\,t}\,\mathrm{Q}\,(t) = k\,\mathrm{Q}\,(t)$

```
(%i2)  desolve(diff(Q(t),t)=k*Q(t),Q(t));
```

(%o2) $\mathrm{Q}\,(t) = \mathrm{Q}\,(0)\,e^{k\,t}$

Next we will analyze one particular example of an exponential growth from nature, the growth of a snail shell.

3 Exponential spiral

Here we will use the feature of the GeoGebra's drawing pad with its geometrical tools and sliders to compare the real curve on a photo with the curve drawn in the program. The parameters of the drawn curve will be controlled by sliders in order to merge it with the real curve as closely as possible. This method is inspired by [2]. Although it is well known and used, its presented application to the exponential spiral by means of specific GeoGebra features represents, as far as we know, an original approach to the problem.

The growth of a snail is a very nice and descriptive example of an exponential process with an indisputable benefit; in contrast to the growth of a quantity of bacteria in food, decay of a radioactive isotope in a material etc., it is clearly recognizable from the shape of a snail's shell and indeed its investigation is absolutely safe.
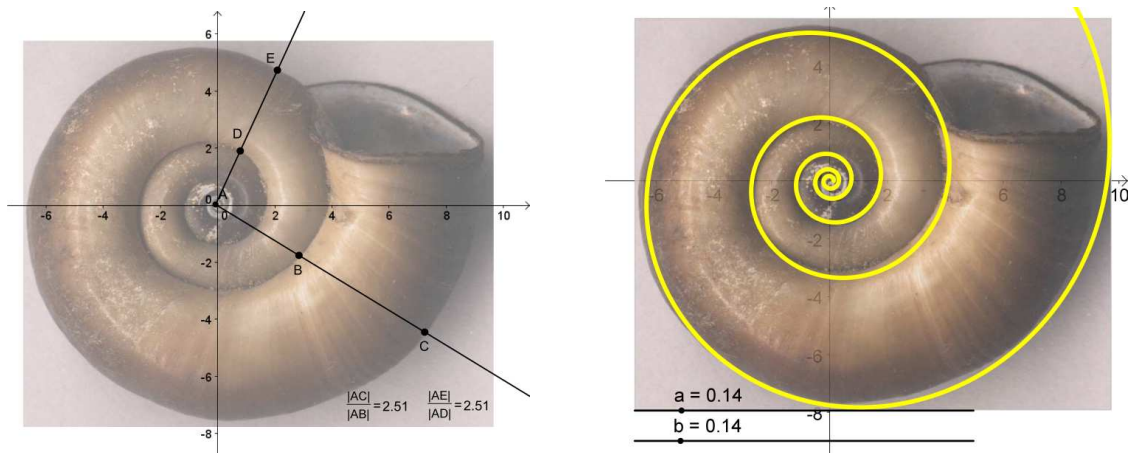
a) Ratios of radial coordinates, [25]   b) Parameters of the spiral, [26]

Figure 2: Planorbarius corneus' shell (Photo by Michal Maňas [1], processed by [17].)

All we need to study a snail's growth is a photo of its shell in some of the most common image file format (for example, with the extensions gif, jpg or bmp). We simply insert it on the drawing pad of the GeoGebra software and adjust it to the origin of its coordinate system. Then we can take advantage of GeoGebra's tools to examine qualities of the pictured shell's spiral. We can propose the hypothesis that the growth of a snail's body and consequently its shell follows the same rule as the growth of a bank deposit or a bacterial colony mentioned above. The bigger the body the higher the rate of change of its proportions, e.g. cross-section, in time.

Be aware of the fact that the method we will use is not a rigorous scientific method. Its result depends for example on the screen resolution, proportions of the picture, its position and a user's patience with merging the real curve with its representation. Moreover, the information value of the result will depend on the type of photographed snail. Because of the two-dimensional characteristic of a photo we should use a snail that has a shell that is as planar as possible to avoid the distortion caused by the nonuniform distribution of the shall's volume in the third-hidden dimension. The shell of the common freshwater snail "Planorbarius corneus", which is used in Figure 2, has such qualities.

Another almost perfect logarithmic spiral can be found for example in the photo of the cross-sectioned nautilus shell by the American photographer Edward Weston [12, 13].

The key to the proof of the exponential growth of the snail's shell is hidden in the relation between the shape of its spiral and the graph of the exponential growth, an example of which is shown in Figure 1 b). To reveal this relation we replace the cartesian coordinates (x, y) with the polar coordinates (r, φ), specifically the x coordinate with the angular coordinate φ and the y coordinate with the radial coordinate r. Then, if our hypothesis is correct, we deal with the so-called logarithmic spiral (also called exponential spiral) [16] the equation of which has in the polar coordinates the form

$$r(\varphi) = ae^{b\varphi}, \tag{5}$$

where $a, b \in \mathbb{R}$ are parameters that control the shape of a spiral.

This possible exponential nature of a snail's growth can be verified by the test of the invariance of the ratio of the radial coordinates of two points tracing the shell's spiral with the same angular coordinate φ, irrespective of its size (see Figure 2 a)). This characteristic, which can be expressed by the formula

$$\frac{e^{\varphi+2\pi}}{e^{\varphi}} = e^{2\pi} = \text{const.}, \tag{6}$$

is the "polar" variant of a typical quality of exponential functions to transform an arithmetic progression of the independent variable x values into a geometric progression of the corresponding dependent values e^x (or ae^{bx} to be more general). For illustration consider three values x_1, x_2, x_3 that are members of the arithmetic progression with the difference d, that is $x_1, \ x_2 = x_1 + d, \ x_3 = x_2 + d$. Then the corresponding values of the exponential function $y(x) = e^x$ form the geometric progression with the quotient e^d as follows from the formulas

$$e^{x_1}, \ e^{x_2} = e^{x_1+d} = e^{x_1}e^d, \ e^{x_3} = e^{x_2+d} = e^{x_2}e^d. \tag{7}$$

Thus the ratio of two consecutive values from (7) is constant:

$$\frac{e^{x_3}}{e^{x_2}} = \frac{e^{x_2}}{e^{x_1}} = e^d = \text{const.} \tag{8}$$

To verify (6) we use the GeoGebra tools *Ray through Two Points* (see the rays AC and AE in Figure 2 a)), *New Point* (see the points B and D in Figure 2 a), [26]) and function *Distance* to measure the distances of two points of the shell's curve with the common ray to the pole. Then we compute the ratio of these distances and display it on the drawing pad, as you can see in Figure 2 a) (the ratios $|AC|/|AB|$ and $|AE|/|AD|$). Then, using the tool *Move* we can move the point defining the ray (C or E in Figure 2 a)) along the curve. We see that the second point also moves along the curve. What does this show? Let us remember that GeoGebra keeps the dividing ratio of three points on a ray (two of them being the ray's defining points) as one of its two defining points is moved. Hence the dividing ratio of the considered three points A, B, C or A, D, E is constant which according to (8) correspond to the exponential characteristic of the spiral. We can confirm that the snail's shell draws the logarithmic spiral with the equation (5).

After this step of verification we can proceed with the determination of parameters a and b from the equation (5) to get the spiral's approximate equation in polar coordinates. The particular values corresponding to the shell's spiral in the photo (see Figure 2) can be found in GeoGebra with the use of two sliders, one to control the values of a, the second to control the values of b (see Figure 2 b)).

The GeoGebra does not enable us to plot the spiral (5) directly in the polar coordinates. It supports plotting only in the Cartesian coordinate system. Therefore, to plot the spiral we must transform its polar representation (5) into the corresponding Cartesian representation. Suppose we have defined the sliders for parameters a and b. First, we define the radial coordinate r as a function of a variable x: `r(x)=a*e^(b*x)`. Then, we use the function `Curve[r(u)·cos(u),r(u)·sin(u),u,0,8·π]` to draw four loops of the corresponding logarithmic spiral. Finally, moving the sliders we change the shape of the spiral until it best merges with the real spiral of the shell on the background of the drawing pad (see Figure 2 (b)). The respective values of a and b, i.e. $a = 0.14$, $b = 0.14$, are what we are searching for. Then the logarithmic spiral in Figure 2 has the equation $r(\varphi) = 0.14e^{0.14\varphi}$. The acquired value 0.14 of b well matches the value $b = \frac{1}{2\pi}\ln 2.51 \doteq 0.146$ derived from the ratios $\frac{|AC|}{|AB|} = \frac{|AE|}{|AD|} = 2.51$ in Figure 2 (a).

4   Falcon's attack trajectory

The logarithmic spiral is also mentioned as the "equiangular spiral" [5, 6]. It means that every line through the pole intersects the spiral at the same angle. This is another unique property of the logarithmic spiral which can also be simply verified by GeoGebra, but this time with the tools *Tangents* and *Angle*. We leave this proof to the reader and will now focus on one particular demonstration of equiangularity of the logarithmic spiral in nature. Mario Livio mentions in [5], on page 120, that the peregrine falcon, one of the fastest birds, traces the logarithmic spiral when it is approaching its prey which it has sighted from a great distance.

This seems to be senseless considering the fact that direct flight would enable the falcon to fully apply its unbeatable speed. Why don't peregrines immediately take the shortest route to their prey? Mario Livio offers an explanation in his book. He mentions the biologist Vance A. Tucker who found and published in [9] and [10] the answer to the given question. He found that the spiral trajectory of the falcon's attack, which characterizes the initial phase of approaching its prey from a great distance, is the result of a compromise between the falcon's desire for fast action and the reality of its eyes' anatomy. The eyes' anatomy enables the use of the eye's most acute vision only in a specific line of sight which deviates considerably from the frontal line [9]. Tucker proved in [10] that to move along the spiral with its head straightened is more effective for the falcon than to move directly with its head turned about 40 degrees to the left or to the right.

4.1   The problem assignment

Modeling and simulation of real-world processes are important tasks in modern mathematics. We would like to show in this paragraph that computers give us a possibility of producing models of real-world phenomena with our students using their standard mathematical knowledge like the Pythagorean theorem, the laws of sines and cosines, properties of congruent triangles etc.

Let's try to model the planar projection of the falcon approaching its prey. Our task will be as follows.

Problem: *Approximate the trajectory of the movement of the falcon during its attack considering the two above mentioned initial conditions: (i) line of sight deviated α degrees to the left or right from the*

frontal direction, i.e. from the direction of the Falcon's instantaneous velocity $\vec{v}$, and (ii) the necessity for the constant visibility of its prey (see Fig. 3 a)).
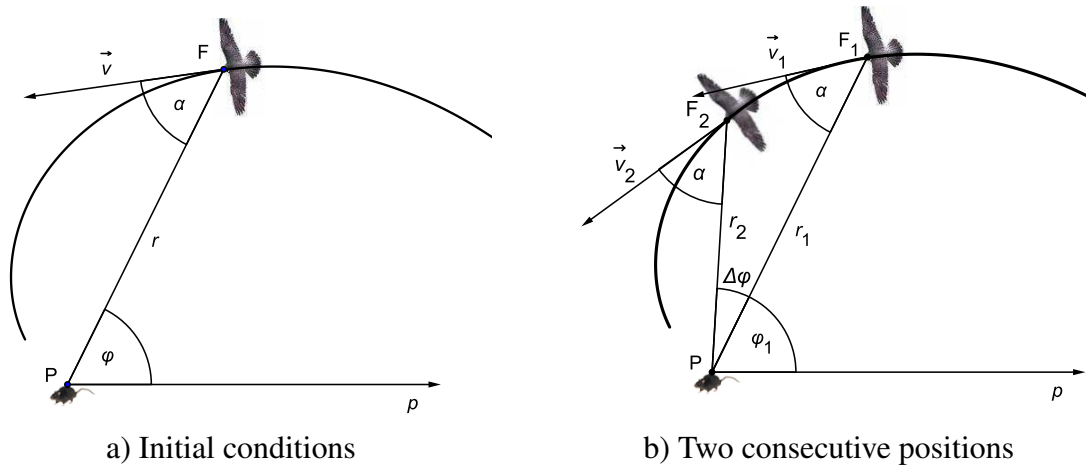

a) Initial conditions   b) Two consecutive positions

Figure 3: Peregrine falcon approaching its prey from a great distance (Plots produced with [17].)

4.2   Theoretical background of the solution

In Figure 3 a) we have the top view of the falcon attacking its prey. To locate points in the projection plane we use the polar coordinates with the pole P placed in the position of the prey, which is considered to be motionless, and the polar axis p with the usual direction of the positive Cartesian coordinates' axis x, as shown in Figure 3 a). We will approximate the curve of the falcon's motion by the numerical computation of the reasonable number of its consecutive positions during its flight, which correspond to equal very small changes $\triangle\varphi$ of the polar angle φ. It is clear, and we will show it using Geogebra on page 227, that a smaller $\triangle\varphi$ is a more accurate approximation of the Falcon's path.

First, we derive the relation between the polar coordinates of two consecutive approximate positions of the falcon. Let its initial position be given by the polar coordinates (r_1, φ_1). We are interested in its subsequent position (r_2, φ_2), where $\varphi_2 = \varphi_1 + \triangle\varphi$ (see Figure 3 b)). As the $\triangle\varphi$ is getting smaller, the arc of the spiral between the two related positions of the falcon is approximated by the secant coming through these two positions. Then we can substitute the wedge F_1F_2P with the triangle F_1F_2P, as shown in Figure 4 a), and derive the r_2 by invoking the law of sines on this triangle. It states that

$$\frac{r_2}{\sin\alpha} = \frac{r_1}{\sin\left(\pi - (\alpha + \triangle\varphi)\right)}, \tag{9}$$

where $\sin\left(\pi - (\alpha + \triangle\varphi)\right) = \sin\left(\alpha + \triangle\varphi\right)$. Thus, (9) can be rewritten to the form

$$\frac{r_2}{\sin\alpha} = \frac{r_1}{\sin\left(\alpha + \triangle\varphi\right)}. \tag{10}$$

Then, we simply isolate r_2 from (10) to get

$$r_2 = r_1 \frac{\sin \alpha}{\sin (\alpha + \triangle\varphi)}. \tag{11}$$

a) $\triangle\varphi$ is very small   b) $\mathrm{d}\varphi$ is infinitesimally small
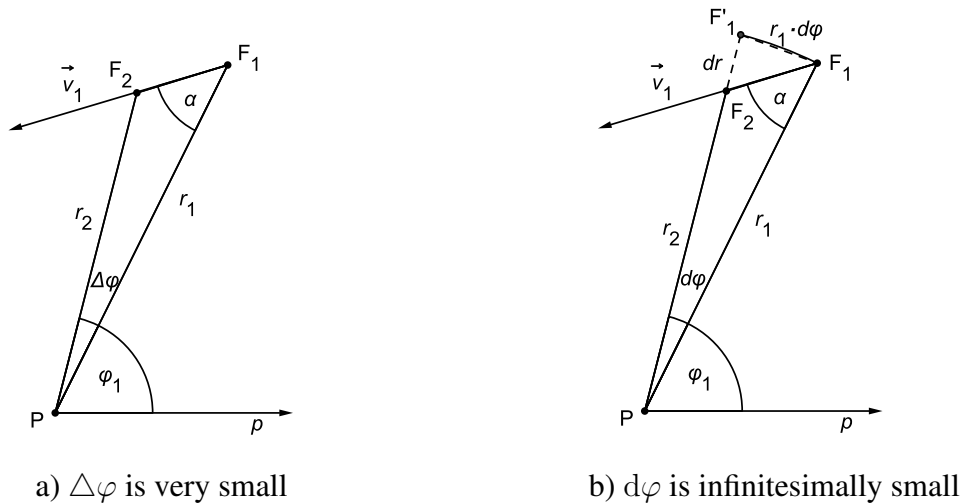
Figure 4: Two consecutive positions of the falcon (Plots produced with [17].)

Therefore, the approximate coordinates (r_2, φ_2) of the falcon's new position, related to the small change $\triangle\varphi$ of the polar angle φ from its initial value φ_1, are given by

$$(r_2, \varphi_2) = \left(r_1 \frac{\sin \alpha}{\sin (\alpha + \triangle\varphi)}, \varphi_1 + \triangle\varphi \right). \tag{12}$$

Then, knowing the values of r_1, φ_1 and $\triangle\varphi$ we can compute the coordinates (r_2, φ_2) according to (12). The same method can be applied to the successive computation of the other approximate positions (r_3, φ_3), (r_4, φ_4), etc. of the Falcon on its path to the prey. We simply iterate the next generalization of (12) that relates two subsequent members of such a progression of positions

$$(r_{n+1}, \varphi_{n+1}) = \left(r_n \frac{\sin \alpha}{\sin (\alpha + \triangle\varphi)}, \varphi_n + \triangle\varphi \right). \tag{13}$$

The resulting coordinate values can be stored in a table and drawn in the plane equipped with the polar coordinate system. GeoGebra is a very convenient tool with which to do so. The result is shown in Figure 5. Let us look in detail at the particular steps taken in solving the problem.

4.3   Solution in GeoGebra

First, we will define four sliders for the control of values r_1, φ_1, α and $\triangle\varphi$, [27]. These sliders are visible near the top of the drawing pad in Fig. 5.

Second, we will open the 'Spreadsheet View' of GeoGebra and complete it in the manner that is illustrated in table 1. We will fill in the first three lines manually with the entries given in the table.

Then, we will copy the content of cells A3, B3, C3 and D3 from the third line into the next several lines, the number of which is given by the number of positions we want to draw. Thanks to the relative addresses of the cells that are used in the formulas (see Table 1) we will get successive iterations of the polar (columns A and B), respective Cartesian (columns C and D) coordinates computed according to (13) in the lines of the table (see also the 'Spreadsheet View' in Figure 5).

	A	B	C	D
1	$\texttt{"}\varphi\texttt{[i]"}$	$\texttt{"r[i]"}$	$\texttt{"x[i]"}$	$\texttt{"y[i]"}$
2	φ_1	r_1	$B2 \cdot \cos{(A2)}$	$B2 \cdot \sin{(A2)}$
3	$A2 + \triangle\varphi$	$B2 \cdot \sin{(\alpha)}/\sin{(\alpha + \triangle\varphi)}$	$B3 \cdot \cos{(A3)}$	$B3 \cdot \sin{(A3)}$
4	$A3 + \triangle\varphi$	$B3 \cdot \sin{(\alpha)}/\sin{(\alpha + \triangle\varphi)}$	$B4 \cdot \cos{(A4)}$	$B4 \cdot \sin{(A4)}$
5	$A4 + \triangle\varphi$	$B4 \cdot \sin{(\alpha)}/\sin{(\alpha + \triangle\varphi)}$	$B5 \cdot \cos{(A5)}$	$B5 \cdot \sin{(A5)}$
6	...	...	...	...

Table 1: Iterative computation by means of the GeoGebra Spreadsheet, [27]

Finally, we will select all the respective data in the columns C and D and create a list of points from them. The list is automatically plotted. It results in a series of points, which, without any doubt, draw a spiral on the 'Drawing pad' as seen in Figure 5.
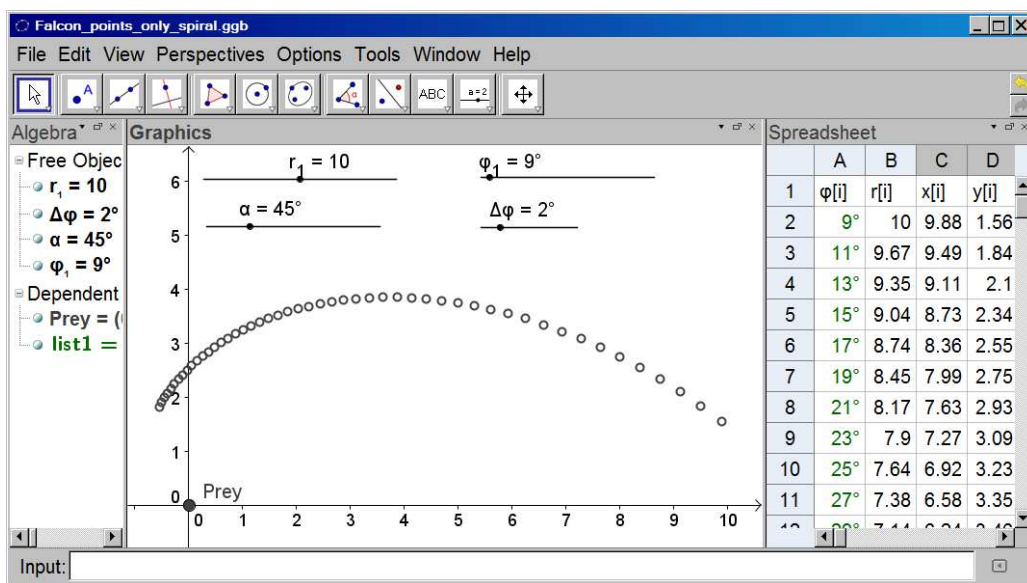

Figure 5: Peregrine falcon approaches its prey along the logarithmic spiral (Solved by use of [17], [27])

4.4 Verification of the numerical solution

The last step of our investigation should be to verify that the curve indicated by the points in Figure 5 satisfies the equation $r(\varphi) = ae^{b\varphi}$ of the logarithmic spiral, where $a, b \in \mathbb{R}$ are parameters that

control its course. We will define sliders for the values of a and b, plot the spiral $r(\varphi) = ae^{b\varphi}$ and by moving the sliders, try to merge it as much as possible with the plotted points, [28]. The success of such verification depends on the accuracy of the falcon's position approximation, i.e. it depends on the smallness of the angle difference $\triangle\varphi$ of two successive positions. In Figure 6 we can see how the logarithmic spiral given by the equation $r(\varphi) = 11.66e^{-0.96x}$ sufficiently merges with the series of approximated positions corresponding to the initial values $\alpha = 45°$, $\triangle\varphi = 2°$, $r_1 = 10$ and $\varphi = 9°$.
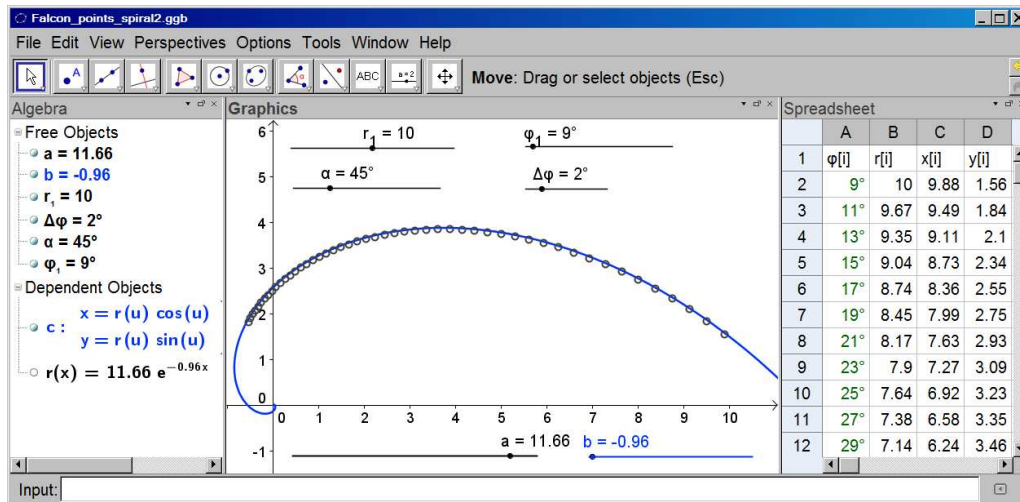

Figure 6: Approximation versus the exact logarithmic spiral (Solved by use of [17], [28])

4.5 Solution in CAS

A computer algebra system (CAS) allows us to use another method for the verification of the exponential quality of the falcon's trajectory. We can simply derive the equation of the trajectory from the given initial conditions. It would be most satisfactory to get the result in the form of the equation $r(\varphi) = ae^{b\varphi}$ with such values of parameters a and b that absolutely fit the path of the approximated positions corresponding to a reasonably small $\triangle\varphi$.

To derive the equation of the falcon's trajectory see the triangle $F_1F_1'P$ in figure 4 b). Consider the $\triangle\varphi$ infinitesimally small. It changes into the differential $\mathrm{d}\varphi$ and the rays PF_1, PF_1' can be considered as parallel. Then $F_1F_2F_1'$ is the right-angled triangle and consequently

$$\tan\alpha = -\frac{r \cdot \mathrm{d}\varphi}{\mathrm{d}r}, \tag{14}$$

where the minus sign compensates the negativity of $\mathrm{d}r$, which is caused by the shortening of the Falcon's radial coordinate r while it is approaching its prey. The formula (14) can be rewritten as

$$\frac{\mathrm{d}r(\varphi)}{\mathrm{d}\varphi} = \frac{-1}{\tan\alpha} \cdot r. \tag{15}$$

If students have not yet undertaken any calculus courses we can solve this differential equation (15) by means of a CAS. See the code of its solution in wxMaxima ([19], [29]):

```
(%i1)  Equation15:'diff(r,phi)=-1/tan(alpha)*r;
```

$$(\%o1) \quad \frac{d}{d\phi} r = -\frac{r}{\tan(\alpha)}$$

```
(%i2)  define(r(phi),rhs(ode2(Equation15,r,phi)));
```

$$(\%o2) \quad \mathrm{r}(\phi) := \%c\, e^{-\frac{\phi}{\tan(\alpha)}}$$

```
(%i3)  alpha:%pi/4$
       float(solve(r(9*%pi/180)=10,%c));
```

$$(\%o4) \quad [\%c = 11.70088787496422]$$

```
(%i5)  %c:11.7$
       r(phi);
```

$$(\%o6) \quad 11.7\, e^{-\phi}$$

After assigning the initial values $\alpha = \dfrac{\pi}{4}$, $r_1 = 10$ and $\varphi = 9\dfrac{\pi}{180}$ into the result (see the line `%i3`) we get the equation of the corresponding logarithmic spiral (see `%o6`): $r(\varphi) = 11.7\cdot e^{-\varphi}$, the parameters' values $a = 11.7$ and $b = -1$ of which are very close to those derived with the help of sliders on page 226 ($a = 11.66$ and $b = -0.96$). Finally, we plot the curve of this equation on the 'Drawing pad' and compare it with the points of the approximated positions, as shown in Figure 7.
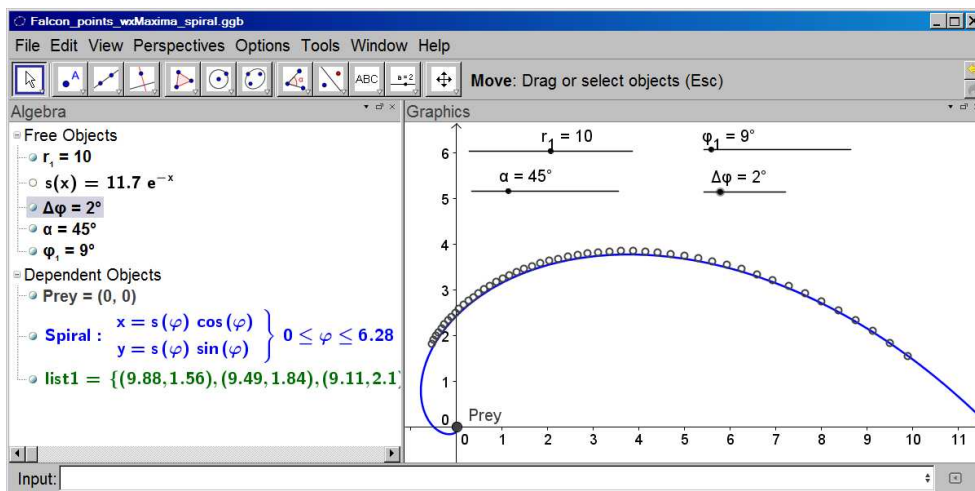

Figure 7: Comparison of the GeoGebra approximation with the spiral given by wxMaxima (Solved by use of [17], [30].)

Comparison of the real computed spiral with the approximated positions gives us a good chance to demonstrate, with the use of a corresponding slider, how the change of $\triangle\varphi$ affects the accuracy of the approximation (see Figure 7).

5   Conclusion

The paper has introduced selected real-world examples the solutions of which can be realized by the proper use of software in the classroom in such a way that is motivating and beneficial to students. Such programs can help students to overcome the possible mathematical complexity of a problem and to penetrate the core idea of the investigated phenomenon. The process of solving such problems in the classroom introduces mathematics as a living and useful science, the application of which crosses boundaries between it and, what at first glance seem distinct disciplines, such as economics, biology, physics, geometry, computer science etc. It gives students a chance to experience the usefulness of the mathematical knowledge they have learnt at school.

The problems that were presented in the paper come from teacher training courses at the University of South Bohemia. The last problem dealing with the falcon's flight was developed by the paper's author. The remaining problems are more or less known but we aimed at the original method of solving them. Our ambition was to show applications of certain features of technology rather than the particular software. The best softwares of today can be bettered in the future. Let us for example mention the development of the new version of GeoGebra 4.2 with CAS built in. Its experimental beta release is already downloadable [18] and its abilities make real the possibility to utilize only GeoGebra instead of its combination with wxMaxima to solve presented problems.

Acknowledgments

References

[1] BioLib.cz [online]. Available at http://www.biolib.cz/cz/taxonimage/id1480/?taxonid=2607 (Under the Creative Commons Attribution License 3.0).

[2] Böhm, J. Background Pictures as Stimulating Means for Math Education. *Conference proceedings of the 3rd Conference on use of computers in mathematics education.* University of South Bohemia in České Budějovice, Czech Republic, 2007, pp. 15–20.

[3] Feynman, R., Leighton, R. & Sands, M. *The Feynman Lectures on Physics.* 2nd ed., Volume 1, Addison Wesley, USA, 2005.

[4] Hejný, M., Kuřina, F. *Dítě, škola a matematika* [A child, school and mathematics. Constructivist approaches to teaching]. 2nd ed., Portál, s.r.o., Praha, Czech Republic, 2009.

[5] Livio, M. *The Golden Ratio: the story of phi, the world's most astonishing number.* 1st ed., Broadway Books, USA, 2003.

[6] Maor, E. *e: The story of a number.* 1st ed., Princeton University Press, USA, 1998.

[7] Mathematics Education - Constructivism in the Classroom. In *The Math Forum @ Drexel* [online]. Available at http://mathforum.org/mathed/constructivism.html.

[8] Paenza, A. *Matematiko, jsi to ty?* ("Matematica...Estas Ahi?" in original). Kniha Zlín, Zlín, Czech Republic, 2010.

[9] Tucker, V. A. The deep fovea, sideways vision and spiral flight paths in raptors. [online] *The Journal of Experimental Biology* 203, p. 3745 - 3754 (2000). Great Britain: The Company of Biologists Limited 2000. Available at http://jeb.biologists.org/content/203/24.

[10] Tucker, V. A. et al. Curved flight paths and sideways vision in peregrine falcons (Falco peregrinus). [online] *The Journal of Experimental Biology* 203, p. 3755 - 3763 (2000). Great Britain: The Company of Biologists Limited 2000. Available at http://jeb.biologists.org/content/203/24.

[11] Weimer, R. C. *Applied Calculus with Technology*. Brooks/Cole Publishing Company, USA, 1998.

[12] Edward Weston. In *Wikipedia: The Free Encyclopedia* [online].
Available at http://en.wikipedia.org/wiki/Edward Weston.

[13] The Dayton art institute. *Edward Weston, a photographer's love in life* [online]. Available at http://www.daytonartinstitute.org/exhibits/weston.html.

[14] Constructivism (learning theory). In *Wikipedia: The Free Encyclopedia* [online].
Available at http://en.wikipedia.org/wiki/Constructivism(learning theory).

[15] Compound interest. In *Wikipedia: The Free Encyclopedia* [online]. Available at http://en.wikipedia.org/wiki/Compound interest.

[16] Logarithmic spiral. In *Wikipedia: The Free Encyclopedia* [online].
Available at http://en.wikipedia.org/wiki/Logarithmic spiral.

Software packages

[17] GeoGebra, free mathematics software for learning and teaching, http://www.geogebra.org.

[18] GeoGebra 4.2 Beta Release (with CAS), experimental version,
http://www.geogebra.org/cms/en/roadmap.

[19] wxMaxima, a document based interface for the computer algebra system Maxima,
http://wxmaxima.sourceforge.net.

Supplemental Electronic Materials

[20] Hašek, R., *Illustration of Euler's number using a spreadsheet*, GeoGebra file, 2011,
Euler-number.html   Euler-number.ggb.

[21] Hašek, R., *Definition of Euler's number as a limit*, wxMaxima worksheet, 2011,
Limit-e.wxm.

[22] Hašek, R., *Maximum balance after n years of saving with continuous compounding (graph)*,
GeoGebra file, 2011, Maximum-balance.html   Maximum-balance.ggb.

[23] Hašek, R., *Maximum balance after n years of saving with continuous compounding (limit)*, wxMaxima worksheet, 2011, Maximum-balance.wxm.

[24] Hašek, R., *Equation of the exponential model*, wxMaxima worksheet, 2011, Exponential-model.wxm.

[25] Hašek, R., *Analysis of the exponential growth of the Planorbarius corneus' shell (a photo on the desktop's background)*, GeoGebra file, 2011, Planorbarius-shell-ratios.html   Planorbarius-shell-ratios.ggb.

[26] Hašek, R., *Spiral of the Planorbarius corneus' shell (a photo on the desktop's background)*, GeoGebra file, 2011, Planorbarius-shell-spiral.html   Planorbarius-shell-spiral.ggb.

[27] Hašek, R., *Peregrine falcon approaches its prey along the logarithmic spiral (numerical solution)*, GeoGebra file, 2011, Falcon-points-only.html   Falcon-points-only.ggb.

[28] Hašek, R., *Peregrine falcon approaches its prey along the logarithmic spiral (confrontation of the numerical solution with the spiral controlled by sliders)*, GeoGebra file, 2011, Falcon-points-spiral.html   Falcon-points-spiral.ggb.

[29] Hašek, R., *The falcon's trajectory computation with wxMaxima*, wxMaxima worksheet, 2011, Falcon-points-wxMaxima-spiral.wxmx.

[30] Hašek, R., *Peregrine falcon approaches its prey along the logarithmic spiral (comparison of the numerical solution with the solution found by wxMaxima)*, GeoGebra file, 2011, Falcon-points-wxMaxima-spiral.html   Falcon-points-wxMaxima-spiral.ggb.